Monopole and Berry Phase in Momentum Space in Noncommutative Quantum Mechanics

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To build genuine generators of the rotations group in noncommutative quantum mechanics, we show that it is necessary to extend the noncommutative parameter θ to a field operator, which one proves to be only momentum dependent. We find consequently that this field must be obligatorily a dual Dirac monopole in momentum space. Recent experiments in the context of the anomalous Hall effect provide evidence for a monopole in the crystal momentum space. We suggest a connection between the noncommutative field and the Berry curvature in momentum space which is at the origine of the anomalous Hall effect.

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A natural generalization of Quantum Mechanics involving noncommutative space time coordinates was originally introduced by Snyder [1] as a short distance regularization to improve the problem of infinite self energies inherent in a Quantum Field Theory. Due to the advent of the renormalization theory this idea was not very popular until A. Connes [2] analyzed Yang Mills theories on noncommutative space. More recently a correspondence between a non-commutative gauge theory and a conventional gauge theory was introduced by Seiberg and Witten [3]. Noncommutative gauge theories were also found as being naturally related to string and M-theory [4].

In this framework an antisymmetric $\theta^{\mu\nu}$ parameter usually taken to be constant [5, 11] is introduced in the commutation relation of the coordinates in the space time manifold $[x^{\mu}, x^{\upsilon}] = i\theta^{\mu\nu}$. This relation leads to the violation of the Lorentz symmetry, a possibility which is intensively studied theoretically and experimentally [12]. Applications of noncommutative theories were also found in condensed matter physics, for instance in the Quantum Hall effect [13**?**] and the non-commutative Landau problem [15, 16, 17] i.e., a quantum particle in the noncommutative plane, coupled to a constant magnetic field with a constant selected θ parameter as usual.

In this letter, we generalize the quantum mechanics in non-commutative geometry by promoting the θ parameter with a new field obeying its own field equations. Note that some authors, (for example [18]) introduced a position dependent θ field using a Kontsevich product [19] in the study of gauge theory. Contrary to these approaches we find that the θ field must be momentum dependent.

The physical motivations of our work are twofold:

(i) For a constant θ field, we show that a quantum particle in a harmonic potential has a behavior similar to a particle in a constant magnetic field θ in standard quantum mechanics, since a paramagnetic term appears in the Hamiltonian. Moreover the particle in the presence of the θ field acquires an effective dual mass in the same way that an electron moving in a periodic potential in solid state physics. Thus it is legitimate to interpret this field like a field having properties of the vacuum. In this context it is natural to extend the theory to a non-constant field. This proposal is strongly enforced by the lack of rotation generators in noncommutative space with a constant θ parameter, i.e. the angular momentum does not satisfy the usual angular momentum algebra. We then show that this θ field is only momentum dependent and that the requirement of the angular momentum algebra, that is the existence of an angular momentum, necessarily imposes a dual Dirac monopole in momentum space field configuration. Thereafter we will intensely use the concept of duality between the quantities defined in momentum space compared with those defined in the position space.

(ii) The second motivation comes for recent theoretical works [20] concerning the anomalous Hall effect in two-dimensional ferromagnets predicting topological singularity in the Brillouin zone, but especially very recent experiments carried out in the same context [21] where a monopole in the crystal momentum space seems to have been discovered. This monopole being a singular configuration of the Berry curvature it appears naturally in the expression of the Hall conductivity [22]. We will consider this framework as a physical realization of our more general theory, where the Berry curvature corresponds to our $\theta(p)$ field.

Consider a quantum particle of mass m whose coordinates satisfy the deformed Heisenberg algebra

$$\left[x^i, x^j\right] = i\hbar q_\theta \theta^{ij}(\mathbf{x}, \mathbf{p}) \ ,$$

$$\left[x^i, p^j\right] = i\hbar \delta^{ij} \ ,$$

$$\left[p^i, p^j\right] = 0,$$

where θ is a field which is a priori position and momentum dependent and q_θ is a charge characterizing the intensity of the interaction of the particle and the θ field. Note that we do not consider any external magnetic field in this work, but its taking into account does not pose a

problem. It is well known that these commutation relations can be obtained from the deformation of the Poisson algebra of classical observable with a provided Weyl-Wigner-Moyal product [23] expanded at the first order in θ.

The following Jacobi identity

$$\left[p^i,\left[x^j,x^k\right]\right]+\left[x^j,\left[x^k,p^i\right]\right]+\left[x^k,\left[p^i,x^j\right]\right]=0, \qquad (1)$$

implies the important property that the θ field is position independent

$$\theta^{jk}=\theta^{jk}(\mathbf{p}). \qquad (2)$$

Then one can see the θ field like a dual of a magnetic field and q_θ like a dual of an electric charge. The fact that the field is homogeneous in space is an essential property for the vacuum. In addition, one easily see that a particle in this field moves freely, that is, the vacuum field does not act on the motion of the particle in the absence of an external potential. The effect of the θ field is manifest only in presence of a position dependent potential.

To look further at the properties of the θ field consider the other Jacobi identity

$$\left[x^i,\left[x^j,x^k\right]\right]+\left[x^j,\left[x^k,x^i\right]\right]+\left[x^k,\left[x^i,x^j\right]\right]=0, \qquad (3)$$

giving the equation of motion of the field

$$\frac{\partial\theta^{jk}(\mathbf{p})}{\partial p^i}+\frac{\partial\theta^{ki}(\mathbf{p})}{\partial p^j}+\frac{\partial\theta^{ij}(\mathbf{p})}{\partial p^k}=0, \qquad (4)$$

which is the dual equation of the Maxwell equation $div\overrightarrow{B}=0$. As we will see later, equation (4) is not satisfied in the presence of a monopole and this will have important consequences.

Now consider the position transformation

$$X^i=x^i+q_\theta a_\theta^i(\mathbf{x},\mathbf{p}), \qquad (5)$$

where a_θ is a priori position and momentum dependent, that restores the usual canonical Heisenberg algebra

$$\left[X^i,X^j\right]=0,$$

$$\left[X^i,p^j\right]=i\hbar\delta^{ij},$$

$$\left[p^i,p^j\right]=0.$$

The second commutation relation implies that a_θ is position independent, while the commutation relation of the positions leads to the following expression of θ in terms of the dual gauge field a_θ

$$\theta^{ij}(\mathbf{p})=\frac{\partial a_\theta^i(\mathbf{p})}{\partial p^j}-\frac{\partial a_\theta^j(\mathbf{p})}{\partial p^i}, \qquad (6)$$

which is dual to the standard electromagnetic relation in position space.

In order to examine more in detail the properties of this new field, let us consider initially the case of a constant field what is usual in noncommutative quantum mechanics. In the case of an harmonic oscillator expressed in terms of the original coordinates $(\mathbf{x},\mathbf{p})$ the Hamiltonian reads

$$H_\theta(\mathbf{x},\mathbf{p})=\frac{\mathbf{p}^2}{2}+\frac{k}{2}x^2, \qquad (7)$$

from which we get: $p^i=m\overset{\cdot}{x}^i-kq_\theta\theta^{ij}x_j$, $\overset{\cdot}{p}^i=-kx^i$ and the equation of motion

$$m\overset{\cdot\cdot}{x}^i=kq_\theta\theta^{ij}\overset{\cdot}{x}_j-kx^i, \qquad (8)$$

which corresponds formally to a particle in a harmonic oscillator submitted to an external constant magnetic field. From equation (6) we deduce that $a_\theta^i(\mathbf{p})=q_\theta\;\theta^{ij}p_j$, so $X^i=x^i+\frac{1}{2}q_\theta\theta^{ij}p_j$, and the Hamiltonian can then be written

$$H_\theta(\mathbf{X},\mathbf{p})=\frac{\left(m_*^{-1}\right)^{ij}p_ip_j}{2}+\frac{k}{2}\mathbf{X}^2-k\frac{q_\theta}{2m}\overrightarrow{\Theta}.\overrightarrow{\mathcal{L}}, \qquad (9)$$

with $\theta^{ij}=\varepsilon^{ijk}\Theta_k$, $\mathcal{L}^i(\mathbf{X},\mathbf{p})=\frac{1}{2}\varepsilon^i{}_{jk}\left(X^jp^k+p^kX^j\right)$ and $\sigma^{ij}=\delta^{ij}\mathbf{\Theta}^2-\Theta^i\Theta^j$, the dual tensor of the Maxwell constraint tensor. Note that the interaction with the field θ is due to the presence of the position dependent harmonic potential and leads to a dual paramagnetic interaction which could be experimentally observable. Like in solid state physics of an electron in the effective periodic potential of the ions, the particle in the θ field acquires an effective mass tensor $m_*^{ij}=m\left(\delta^{ij}+\frac{\hbar^2kq_\theta^2}{4}\sigma^{ij}\right)^{-1}$ which breaks the homogeneity of space. This strong analogy with the vacuum of the solid state leads us to regard this field as a property of the vacuum.

Consider now the problem of angular momentum. It is obvious that the angular momentum expressed according to the canonical coordinates satisfies the angular momentum algebra however it is not conserved

$$\frac{d\overrightarrow{\mathcal{L}}(\mathbf{X},\mathbf{p})}{dt}=kq_\theta\overrightarrow{\mathcal{L}}\wedge\overrightarrow{\Theta}. \qquad (10)$$

In the original (x,p) space the usual angular momentum $L^i(\mathbf{x},\mathbf{p})=\varepsilon^i{}_{jk}x^jp^k$, does not satisfy this algebra. So it seems that there are no rotation generators in the (x,p) space. We will now prove that a true angular momentum can be defined only if θ is a non constant field.

From the definition of the angular momentum we deduce the following commutation relations

$$[x^i,L^j]=i\hbar\varepsilon^{ijk}x_k+i\hbar q_\theta\varepsilon^j{}_{kl}p^l\theta^{ik}(\mathbf{p}),$$

$$[p^i,L^j]=i\hbar\varepsilon^{ijk}p_k,$$

$$[L^i,L^j]=i\hbar\varepsilon^{ij}{}_kL^k+i\hbar q_\theta\varepsilon^i{}_{kl}\varepsilon^j{}_{mn}p^lp^n\theta^{km}(\mathbf{p}),$$

showing in particular that the $sO(3)$ Algebra is broken. To restore the angular momentum algebra consider the transformation law

$$L^i \to \mathbb{L}^i = L^i + M^i_\theta(\mathbf{x},\mathbf{p}), \tag{11}$$

and require the usual algebra

$$[x^i, \mathbb{L}^j] = i\hbar\varepsilon^{ijk}x_k,$$

$$[p^i, \mathbb{L}^j] = i\hbar\varepsilon^{ijk}p_k,$$

$$[\mathbb{L}^i, \mathbb{L}^j] = i\hbar\varepsilon^{ijk}\mathbb{L}_k. \tag{12}$$

The second equation implies the position independent property

$$M^j_\theta(\mathbf{x},\mathbf{p}) = M^j_\theta(\mathbf{p}), \tag{13}$$

while the third leads to

$$M^i_\theta(\mathbf{p}) = \frac{1}{2}q_\theta\varepsilon_{jkl}p^ip^l\theta^{kj}(\mathbf{p}). \tag{14}$$

Putting this equation in (12) we are led to a dual Dirac monopole [25] defined in momentum space

$$\overrightarrow{\Theta}(\mathbf{p}) = \frac{g_\theta}{4\pi}\frac{\overrightarrow{p}}{\mathbf{p}^3}, \tag{15}$$

where we introduced the dual magnetic charge g_θ associated to the Θ field. Consequently we have

$$\overrightarrow{M}_\theta(\mathbf{p}) = -\frac{q_\theta g_\theta}{4\pi}\frac{\overrightarrow{p}}{\mathbf{p}}, \tag{16}$$

which is the dual of the famous Poincare momentum introduced in positions space [26, 27]. Then the generalized angular momentum

$$\overrightarrow{\mathbb{L}} = m\left(\overrightarrow{r}\wedge\overrightarrow{p}\right) - \frac{q_\theta g_\theta}{4\pi}\frac{\overrightarrow{p}}{\mathbf{p}}, \tag{17}$$

is a genuine angular momentum satisfying the usual algebra. It is the summation of the angular momentum of the particle and of the dual monopole field. One can check that it is a conserved quantity.

The duality between the monopole in momentum space and the Dirac monopole is due to the symmetry of the commutation relations in noncommutative quantum mechanics where $\left[x^i, x^j\right] = i\hbar q_\theta\varepsilon^{ijk}\Theta_k(\mathbf{p})$ and the usual quantum mechanics in a magnetic field where $\left[v^i, v^j\right] = i\hbar q\varepsilon^{ijk}B_k(\mathbf{x})$. Therefore the two gauge fields $\Theta(\mathbf{p})$ and $B(\mathbf{x})$ are dual to each other.

Note that in the presence of the dual monopole the Jacobi identity (3) fails:

$$\left[x^i,\left[x^j,x^k\right]\right] + \left[x^j,\left[x^k,x^i\right]\right] + \left[x^k,\left[x^i,x^j\right]\right] = -q_\theta\hbar^2\frac{\partial\Theta^i(\mathbf{p})}{\partial p_i} = -4\pi q_\theta\hbar^2 g_\theta\delta^3(\mathbf{p}). \tag{18}$$

One can interpret this by analogy with the explanation given by Jackiw [28] of a comparable violation of the Jacobi identity between momentum by the Dirac monopole in standard quantum mechanics: the presence of the monopole in momentum space is related to the breaking of the translations group of momentum. As a consequence the addition law of momentum is different from the usual Galilean additional law. Indeed if we define the element of the translations group of momentum by $T(\mathbf{b}) = \exp\left(i\overrightarrow{r}.\overrightarrow{b}/\hbar\right)$, we have the following relation

$$T(\mathbf{b}_1)T(\mathbf{b}_2) = \exp\left\{i\frac{q_\theta}{\hbar}\Phi\left(\mathbf{p};\mathbf{b}_1,\mathbf{b}_2\right)\right\}T(\mathbf{b}_1+\mathbf{b}_2), \tag{19}$$

where $\Phi\left(\mathbf{p};\mathbf{b}_1,\mathbf{b}_2\right)$ is the flux of Θ through a triangle with three tops located by the vectors : $\overrightarrow{p}$, $\overrightarrow{p}+\overrightarrow{b_1}$, and $\overrightarrow{p}+\overrightarrow{b_1}+\overrightarrow{b_2}$. This term is responsible for the violation of the associativity which is only restored if the following quantification equation is satisfied

$$\int d^3p\frac{\partial\Theta^i}{\partial p_i} = \frac{2\pi n\hbar}{q_\theta} \tag{20}$$

leading to $q_\theta g_\theta = \frac{n\hbar}{2}$, in complete analogy with Dirac's quantization [28].

It is interesting to mention that singular configuration in momentum space, seems to have been discovered in the very beautiful experiments of Fang and al. [21] in the context of the anomalous Hall effect in a ferromagnetic crystal. The strong analogy between this result and the monopole we deduced from symmetry consideration in noncommutative quantum mechanics, suggest us interpreting their Berry curvature in the AHE as our noncommutative field. The main point is the consideration of the Berry phase

$$a^\mu_n(\mathbf{k}) = i\left\langle u_{n\mathbf{k}}\right| d_k \left|u_{n\mathbf{k}}\right\rangle$$

where the wave function $u_{n\mathbf{k}}(x)$ are the periodic part of the Bloch waves. In their work, the authors introduced a gauge covariant position operator of the wave packet associated to an electron in the n band

$$x^\mu = i\frac{\partial}{\partial k_\mu} - a^\mu_n(\mathbf{k}), \tag{21}$$

whose commutator is given by

$$[x^\mu, x^\nu] = \frac{\partial a^\nu_n(\mathbf{k})}{\partial k^\mu} - \frac{\partial a^\mu_n(\mathbf{k})}{\partial k^\nu} = -iF^{\mu\nu}(\mathbf{k}) \tag{22}$$

where $F^{\mu\nu}(\mathbf{k})$ is the Berry curvature in momentum space.

The connection with our noncommutative quantum mechanics theory is then clearly apparent. The $\theta(\mathbf{p})$ field corresponds to the Berry curvature $F(k)$ and $a_\theta(\mathbf{p})$ is associated to the Berry phase $a_n(k)$. This shows that physical situations with a Berry phase living in momentum space could be expressed in the context of a noncommutative quantum mechanics. Of course this formal analogy requires more work to deepen the relation between the noncommutative quantum mechanics formalism and the Berry phase in momentum space.

Our work is justified by the will to preserve exact symmetries. For that we found the necessity to promote the θ parameter of the noncommutative quantum mechanics to a $\theta(\mathbf{p})$ field. Then we showed that the restoration of the Heisenberg algebra implies the existence of a dual gauge field in momentum space. We proved that configuration of the field which makes it possible to build an angular momentum which satisfies the $sO(3)$ algebra and which is preserved, is a dual monopole in momentum space. This monopole is responsible for the violation of the Jacobi identity and implies the non associativity of the law of addition of the momentum. To restore associativity a Dirac's quantization of the dual charges is necessary. As a physical realization of our theory we can interpret the $\theta(\mathbf{p})$ field as a Berry curvature associated to a Berry phase expressed in momentum space in the context of the anomalous Hall effect.

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